Processor Sharing and Pricing Implications

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***Abstract*—We study pricing models for bandwidth sharing that do not depend on detailed statistical knowledge of network traffic. For a multiclass, processor sharing server, we show that three common pricing models, namely fixed rate pricing, Vickrey-Clarke-Groves pricing, and congestion-based pricing, are linked to the zeroth, first, and second moments, respectively, of the number of users in the system. We derive expressions for these quantities and provide insights into the operator's revenue and user payments.**

I. Introduction

The processor sharing discipline, where a server allocates an equal share of the resource to each user in the system, finds applications in a myriad of domains such as bandwidth allocation in networks and in multi-programming computer systems (and data-centers). A key benefit of the processor sharing discipline is that it results in insensitivity, i.e., the stationary distribution of the number of users is characterized by the first order statistics of the arrival process and the service distribution. Insensitive allocations also exhibit product-form distribution. In the case of a single server, processor sharing is the socially optimum policy when the users' utilities are the logarithm of their allocated bandwidth.

The paper explores fixed rate pricing, Vickrey-Clarke-Groves (VCG) based pricing, and congestion-based pricing (or Lagrange shadow prices) for charging users that are allocated resources via processor sharing. The prices are a function of the load (the number of users present) on the system. We show that the mean user payments and operator revenue can be obtained from the zeroth, first, and second order statistics of the number of users in the system. The paper also explores two implementation mechanisms for the three pricing models: the post-payment mechanism where the users pay the exact charge accrued during their service and the pre-payment mechanism where the operator charges the user based on the expected sojourn time and the perceived load on the system.

The processor sharing discipline, although simple, is of key interest since the insensitivity property offers *predictability* in the expected revenue to the operator and in the expected payments by the user. This allows us to gain insights into the structure of the pricing models. Motivated by this, we explicitly derive the expression for the second moment of the number of users and the correlation between the numbers of users of two different classes in a processor sharing environment, a contribution that is of independent interest in the study of insensitive allocations. For a further discussion on processor sharing and insensitivity, the reader is directed to [1], [2], [3], [4] and the references therein.

II. System Model

Consider a single server with M/G inputs, capacity C, and using the processor sharing discipline. The system consists of users that represent file transfers or flows. Each user arriving to the server belongs to one of K classes, indexed by the set $\{1, \ldots, K\}$. A class is distinguished by its arrival and service requirement characteristics. Class k user arrivals are modeled as a Poisson process with rate λ_k. Each class k user brings a random amount of work, independent and identically distributed, with a common general distribution with mean ν_k. At an instant t, let $\vec{x}(t) := (x_1(t), \ldots, x_K(t))$ denote the number of users of each type present in the system with $x_k(t) \geq 0$ denoting the number of class k users. Let $|\vec{x}|$ denote the ℓ_1-norm of $\vec{x}$. The allocation to a user in the system is $C/|\vec{x}|$ and the total allocation to class k is given by $\Lambda_k(\vec{x}) = x_k C/|\vec{x}|$. Let $\vec{\Lambda}(\vec{x}) = (\Lambda_1(\vec{x}), \ldots, \Lambda_K(\vec{x}))$. As mentioned in the previous section, processor sharing results from maximizing social welfare when each user has a $\log$ utility function. The $\log$ utility function is also of interest since its solution coincides with the Nash bargaining solution and since it is the only scale invariant utility function. We will assume this utility function when we derive the payments under VCG based pricing and congestion-based pricing.

Define $\alpha_k := \lambda_k \nu_k$ and $\vec{\alpha} := (\alpha_1, \ldots, \alpha_K)$. The traffic intensity of class k is denoted by $\rho_k = \alpha_k / C$. Let the total traffic intensity, ρ, be given by $\rho = \sum_{k=1}^{K} \rho_k$. We use the notation $\vec{\alpha}^{\vec{x}} = \prod_{k=1}^{K} \alpha_k^{x_k}$ for convenience.

Let π be the stationary distribution of the underlying Markov process, $\mathbb{P}_N$ be the Palm probability associated with any stationary point process N, and $\mathbb{E}_N$ be the expectation with respect to the Palm probability, which in this case, is the same as the stationary measure due to the Poisson Arrivals See Time Averages property (PASTA). With a slight abuse of notation, let $\mathbb{E}_{\vec{x}}$ be the expectation conditioned on the arrival state $\vec{x}$. Define $\chi(\vec{x}) := \Phi(\vec{x})\vec{\alpha}^{\vec{x}}$ to be an invariant distribution under π where $\Phi(\vec{x})$ is the balance function satisfying

$$\Phi(\vec{x}) = \frac{1}{C} \sum_{m=1}^{K} \Phi(\vec{x} - \vec{e}_m),$$

and $\vec{e}_m$ is a K-dimensional unit vector with a 1 at the m^{th} element (see [2], [4] for a discussion). The stationary distribution $\pi(\vec{x})$ is then given by

$$\pi(\vec{x}) = \frac{\chi(\vec{x})}{\sum_{\vec{y}} \chi(\vec{y})}. \tag{1}$$

III. Performance Metrics

As we will show in the next section, the zeroth, first, and the second moments of the number of users in the system will play an important role in the revenue collected by the operator. We thus start by providing a characterization of these moments, which is of independent interest in the study of processor sharing models. We use the following quantities in our derivations.

$$t(n) = \sum_{\vec{x}:|\vec{x}|=n} \chi(\vec{x}) \tag{2}$$

$$s_k(n) = \sum_{\vec{x}:|\vec{x}|=n} x_k \chi(\vec{x}) \tag{3}$$

$$\bar{s}_k(n) = \sum_{m>n} s_k(m) \tag{4}$$

$$s_{i,j}(n) = \sum_{\vec{x}:|\vec{x}|=n} x_i x_j \chi(\vec{x}) \tag{5}$$

The above terms will be useful in evaluating different moments of the number of users in the system. For example,

$$\sum_{\vec{x}} x_k \chi(\vec{x}) = \sum_{n=0}^{\infty} s_k(n) \text{ and } \mathbb{E}[|\vec{x}|^2 x_i] = \frac{1-\rho}{\Phi(\vec{0})} \sum_{n=0}^{\infty} n^2 s_i(n).$$

The following lemmas evaluate $t(n), s_k(n), \bar{s}_k(n)$, and $s_{i,j}(n)$. The proofs of these and subsequent results are provided in the appendix.

Lemma 1. *Let $t(n)$ be defined as in (2). Then, $t(n) = \Phi(\vec{0})\rho^n$ and $\sum_{n=0}^{\infty} t(n) = \sum_{\vec{y}} \chi(\vec{y}) = \frac{\Phi(\vec{0})}{1-\rho}$.*

Lemma 2. *Let $s_k(n)$ and $\bar{s}_k(n)$ be defined as in (3) and (4). Then*

$$s_k(n) = n\rho^{n-1}\rho_k \Phi(\vec{0}),$$

and

$$\bar{s}_k(n) = \frac{\Phi(\vec{0})}{1-\rho} \rho^n \rho_k \left(n + \frac{1}{1-\rho}\right).$$

Lemma 3. *Let $s_{i,j}(n)$ be defined as in (5). Then,*

$$s_{i,j}(n) = \begin{cases} n(n-1)\rho_i\rho_j\rho^{n-2}\Phi(\vec{0}) & \text{if } i \neq j \\ n\left((n-1)\rho_i^2 + \rho_i\rho\right)\rho^{n-2}\Phi(\vec{0}) & \text{if } i = j. \end{cases} \tag{6}$$

The above three lemmas compute the zeroth, the first, and the second moment of the number of users in the system under the invariant distribution $\chi(\vec{x})$. Note that Lemma 1 gives the expression for the normalizing term in (1) for obtaining the stationary distribution from the invariant distribution. We also define the following terms which are needed for evaluating prices.

$$u(n) := \sum_{\vec{x}:|\vec{x}|>n} (|\vec{x}| - 1/2)\, \pi(\vec{x}) \tag{7}$$

$$v(n) := \sum_{\vec{x}:|\vec{x}|=n} |\vec{x}|^2 \chi(\vec{x}) = n^2 t(n) \tag{8}$$

$$g_k(n) := \sum_{\vec{x}:|\vec{x}|=n} \frac{x_k}{|\vec{x}|} \chi(\vec{x}) = s_k(n)/n \tag{9}$$

Proposition 1. *Let $u(n)$ be defined as in (7). Then,*

$$u(n) = \rho^{n+1}\left(n + \frac{1}{1-\rho} - \frac{1}{2}\right).$$

The proposition is useful for evaluating revenue under VCG pricing.

IV. Applications to Pricing

Having derived expressions for the moments of the number of users in processor sharing, we now discuss their application in the context of three pricing models: fixed rate pricing, VCG-based pricing, and congestion-based pricing. We first establish some notation and preliminary details. Let $R_F(\vec{x})$, $R_V(\vec{x})$, and $R_L(\vec{x})$ respectively denote the revenue per unit-time collected by the operator under fixed rate pricing, under VCG pricing, and under congestion-based pricing, when the number of users is given by $\vec{x}$. We will also discuss revenue collected when a certain Quality of Service (QoS) requirement is imposed (we discuss the particulars of this requirement in Section IV-A). We will let $\bar{R}_F$, $\bar{R}_V$, and $\bar{R}_L$ denote the mean revenue per unit-time collected by the operator under the various pricing models with this QoS constraint. Similarly, let $c_k^F(\vec{x})$, $c_k^V(\vec{x})$, and $c_k^L(\vec{x})$ denote the payment per unit-time by each class k user under the three pricing models in state $\vec{x}$, and $\bar{c}_k^F$, $\bar{c}_k^V$, and $\bar{c}_k^L$ denote the mean payment by each class k user.

A. A QoS Requirement

We will see that under VCG pricing and congestion-based pricing, an operator can collect arbitrarily large revenue by installing small capacity, leading to longer sojourn times and greater accrued payments. To overcome this, we study the use of a QoS requirement defined as follows. A class k user pays the operator only if the rate allocated at time t, $C/|\vec{x}(t)|$, is equal to or greater than r_k. For ease of exposition, all r_k are assumed to be identical, i.e., $r_{\min} = r_k$. The $C/|\vec{x}| \geq r_{\min}$ condition is equivalent to a $|\vec{x}| \leq n^*$ condition where $n^* = \lfloor C/r_{\min} \rfloor$. This provides motivation for the operator to offer sufficient bandwidth allocations.

B. Post-payments vs. Pre-payments

All three pricing models in this paper charge a user based on the number of users in the system. A change in the number of users is reflected in the instantaneous per unit-time price. For a tagged user, the exact charge accrued is evaluated by tracking arrivals and departures during the user's sojourn. In this paper, the mean of this exact payment incurred by a user is derived using sample path arguments from Palm probability. The mean of the operator's revenue is independently derived. Since the total charge accrued is only known at the end of sojourn, we refer to such an implementation as the *post-payment mechanism* or the *post-payment scheme*.

After deriving the mean revenue and post-payment expressions, we also investigate a *pre-payment mechanism* (or scheme) where a user is charged a fee up-front based on the system load and the expected sojourn time observed on arrival. Prices are adjusted to ensure the same mean payment for each class as in the post-payment scheme. A pre-payment scheme has several benefits. First, the user is aware of the payment up-front unlike the post-payment scheme where a user may be billed a large fee caused by sudden high loads during its sojourn. Second, the second moment (and thus the standard deviation) of user payments in the pre-payment scheme can be exactly characterized. Based on the second moment of the fixed rate pricing mechanism and VCG pricing and on our simulations results, we observe that the pre-payment scheme has a smaller second moment, and thus the operator's revenue is predictable with greater confidence.

C. Pricing Models

We implicitly assume that the QoS requirement is always present unless stated otherwise. The first pricing model is *fixed rate pricing* where the user pays a fixed price of β per unit-time per unit-resource, i.e., if a user is allocated Λ resource for time T, the user pays $\Lambda\beta T$. Thus,

$$c_k^F(\vec{x}) = \begin{cases} \frac{\beta C}{|\vec{x}|} & \text{if } 1 \le |\vec{x}| \le n^* \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

The operator's revenue is the aggregate of user payments, i.e.,

$$R_F(\vec{x}) = \begin{cases} \beta C & \text{if } 1 \le |\vec{x}| \le n^* \\ 0 & \text{otherwise.} \end{cases}$$

The $\log$ utility assumption is important for the next two pricing models. Under *VCG pricing*, a user pays the decrease in maximum social welfare caused by it entering the system (see [5]). Let r index over the set of users and with a slight abuse of notation, let Λ_r indicate the allocation to user r. If $|\vec{x}| \ge 2$, this price is calculated as

$$\begin{aligned} c_k^V(\vec{x}) &= \max_{\vec{\Lambda}} \sum_{s \ne r | \Lambda_r = 0} \log(\Lambda_s) - \sum_{s \ne r} \log(\Lambda_s^{PS}) \\ &= (|\vec{x}| - 1) \log \frac{C}{|\vec{x}| - 1} - (|\vec{x}| - 1) \log \frac{C}{|\vec{x}|} \\ &= (|\vec{x}| - 1) \log \frac{|\vec{x}|}{|\vec{x}| - 1}, \end{aligned}$$

where, Λ_s^{PS} is the allocation to user s under processor sharing, i.e., $\Lambda_s^{PS} = C/|\vec{x}|$. The aggregate per unit-time revenue collected by the operator is given by

$$R_V(\vec{x}) = \begin{cases} |\vec{x}|(|\vec{x}| - 1) \log \frac{|\vec{x}|}{|\vec{x}| - 1} & \text{if } 2 \le |\vec{x}| \le n^* \\ 0 & \text{otherwise.} \end{cases}$$

To gain further insights in the revenue and payment problem, the following approximation is shown to hold.

Proposition 2. *$R_V(\vec{x}) \approx |\vec{x}| - \frac{1}{2}$ and the approximation error is $O(1/|\vec{x}|)$. Furthermore, $|\vec{x}| - \frac{1}{2}$ is an upper bound on $R_V(\vec{x})$ for $|\vec{x}| > 0$.*

The above approximation for VCG revenue is used throughout this paper. Thus, the price paid by the user is approximated as

$$c_k^V(\vec{x}) = \begin{cases} \left(1 - \frac{1}{2|\vec{x}|}\right) & \text{if } 2 \le |\vec{x}| \le n^* \\ 0 & \text{otherwise,} \end{cases} \tag{11}$$

and the revenue earned by the operator is given by

$$R_V(\vec{x}) = \begin{cases} |\vec{x}| - 1/2 & \text{if } 2 \le |\vec{x}| \le n^* \\ 0 & \text{otherwise.} \end{cases}$$

In *congestion-based pricing*, the shadow price (or the dual variable of the social welfare maximization problem) is charged, e.g., see [6]. This shadow price has the advantage of leading the system to social welfare in a distributed implementation. The shadow price under processor sharing is $|\vec{x}|/C$. Thus, the payment per unit-time made by a class k user is given by

$$c_k^L(\vec{x}) = \begin{cases} \frac{|\vec{x}|}{C} & \text{if } 1 \le |\vec{x}| \le n^* \\ 0 & \text{otherwise,} \end{cases} \tag{12}$$

and the aggregate revenue collected by the operator is

$$R_L(\vec{x}) = \begin{cases} \frac{|\vec{x}|^2}{C} & \text{if } 1 \le |\vec{x}| \le n^* \\ 0 & \text{otherwise.} \end{cases}$$

Fundamentally, the price per unit-time charged to users under fixed rate pricing, VCG pricing, and the congestion-based pricing are proportional to $1/|\vec{x}|$, to ≈ 1, and to $|\vec{x}|$. The consequence is that a user is charged less (offered a discounted price) at high loads under fixed rate pricing, offered an approximately constant price under VCG pricing, and is charged more (penalized) under congestion-based pricing. We emphasize that demand response, i.e., user behavior reacting to changing price, is not considered in our work.

D. Mean Operator Revenue

The mean operator revenue is derived under the three pricing models. The expressions hold under both pre-payment and post-payment schemes since the mean payment under both schemes is the same from each class. The revenues are per unit-time.

Proposition 3. *The operator's revenue per unit-time under the three pricing models is given by*

$$\bar{R}_F = \beta C \rho (1 - \rho^{n^*}) \tag{13}$$

$$\bar{R}_V = \frac{\rho^2}{2} \left(1 + \frac{2}{1 - \rho}\right) - \rho^{n^* + 1} \left(n^* - \frac{1}{2} + \frac{1}{1 - \rho}\right) \tag{14}$$

$$\bar{R}_L = \frac{1 - \rho}{C} \sum_{n=1}^{n^*} n^2 \rho^n. \tag{15}$$

The proof is provided in the appendix. The proof highlights that the mean revenue earned by the operator for fixed rate pricing, VCG pricing, and congestion-based pricing is respectively related to the zeroth, first, and the second moment of the total number of users in the system, i.e.,

$$\bar{R}_F \propto \mathbb{E}[|\vec{x}|^0 \mathbf{1}_{(1\leq|\vec{x}|\leq n^*)}]$$
$$\bar{R}_V \propto \mathbb{E}[(|\vec{x}|-1/2)\mathbf{1}_{(2\leq|\vec{x}|\leq n^*)}]$$
$$\bar{R}_L \propto \mathbb{E}[|\vec{x}|^2 \mathbf{1}_{(1\leq|\vec{x}|\leq n^*)}].$$

This key insight is attributed to the inherent structure of the pricing models identified in Section IV-C.

E. Post-payments: Exact charge accrued by users

The result on the mean payment by a class k user is presented next.

Proposition 4. *The mean payment by a class k user under the three pricing models is given by*

$$\bar{c}_k^F = \nu_k \beta(1-\rho^{n^*})$$
$$\bar{c}_k^V = \frac{\nu_k}{C}\left(\rho\left(\frac{1-\rho^{n^*}}{1-\rho}\right)+\frac{\rho}{2}-\left(n^*-\frac{1}{2}\right)\rho^{n^*}\right)$$
$$\bar{c}_k^L = \frac{\nu_k(1-\rho)}{C^2}\sum_{n=1}^{n^*} n^2\rho^{n-1}.$$

Note that the metering required by the operator is at the time-scale at which users enter and leave the system.

F. Pre-payments: Freezing Prices on Arrival

In this section, a pre-payment scheme is devised where the user is charged a price up-front on arrival. The price is dependent on the underlying pricing model, i.e., behaves similar to fixed rate pricing, VCG pricing, or congestion-based pricing. However, the price charged to a given user now depends only on the number of users in the system when that user arrives and is based on the expected sojourn time at arrival. The payment is adjusted so that the mean payments by class k users remain the same as in Section IV-E.

Let W_k be the random variable denoting the sojourn time of the class k arrival. Under the pricing model X, where X is a placeholder for F, V, or L, let $\gamma^X(\vec{x})$ be the per unit-time price fixed on class k user's arrival when the arrival observes the system state as $\vec{x}$. Under the pre-payment scheme, the price charged to any class k user is given by

$$p_k^X(\vec{x}) = \gamma_k^X(\vec{x}+\vec{e}_k)\mathbb{E}_{\vec{x}}[W_k]. \quad (16)$$

Proposition 5. *For a processor sharing server with multiclass M/G inputs,*

$$\mathbb{E}_{\vec{x}}[W_k] = A_{k,0}+\sum_{m=1}^{K} A_{k,m}x_m,$$

for some positive coefficients $A_{k,i}$, $0\leq i\leq K$.

The proposition is an immediate consequence of [7, Theorem 6] which shows that the sojourn time of a new arrival can be decomposed into the sum of random variables for each pre-existing user and the new arrival. The proposition follows by taking the expectation of this sum.

Based on the structure of fixed rate pricing, VCG pricing, and congestion-based pricing, we define $\gamma_k^X(\vec{x})$ as

$$\gamma_k^F(\vec{x}) = \sigma_k^F|\vec{x}|^{-1},\ \gamma_k^V(\vec{x}) = \sigma_k^V,\ \text{and}\ \gamma_k^L(\vec{x}) = \sigma_k^L|\vec{x}|.$$

Note that $\gamma_k^X(\vec{x})$ is not zero for $|\vec{x}|>n^*$. The constants σ_k^F, σ_k^V, and σ_k^L are determined by equating the mean payments by class k users to the payments in Section IV-E, i.e.,

$$\mathbb{E}[p_k^X(\vec{x})] = \bar{c}_k^X.$$

Proposition 6. *The constants σ_k^F, σ_k^V, and σ_k^L are*

$$\sigma_k^F = \frac{\nu_k\beta(1-\rho^{n^*})}{(1-\rho)}\left(\frac{A_{k,0}}{\rho}\log\frac{1}{1-\rho} + \frac{1}{\rho^2}\left(\frac{\rho}{1-\rho}-\log\frac{1}{1-\rho}\right)\sum_{m=1}^{K}A_{k,m}\rho_m\right)^{-1}$$

$$\sigma_k^V = \rho(1-\rho^{n^*})+\frac{\rho(1-\rho)}{2}-(1-\rho)\left(n^*-\frac{1}{2}\right)\rho^{n^*}$$

$$\sigma_k^L = \frac{\nu_k(1-\rho)^2}{C^2}\frac{\sum_{n=1}^{n^*}n^2\rho^{n-1}}{A_{k,0}+\frac{2}{1-\rho}\sum_{m=1}^{K}A_{k,m}\rho_m}.$$

The outline of the proof is provided in the appendix.

Since the mean payment by class k remains the same as under Section IV-E, the mean revenue collected by the operator also remains the same as in Section IV-D. The evaluation of the performance metrics in Section III allows the explicit characterization of the second moment (and hence the variation) of class k payments. Define $\mathrm{Li}_2(\rho) = \sum_{n=1}^{\infty}\frac{\rho^n}{n^2}$.

Proposition 7. *The second moment of pre-payments by class k users is given by*

$$\begin{aligned}
\mathbb{E}[(p_k^F(\vec{x}))^2] &= (\sigma_k^F)^2(1-\rho)A_{k,0}^2\frac{\mathrm{Li}_2(\rho)}{\rho}\\
&+(\sigma_k^F)^2(1-\rho)\left(\frac{\rho+3\log(1-\rho)-3\rho\log(1-\rho)}{\rho^3(1-\rho)} + \frac{2\,\mathrm{Li}_2(\rho)}{\rho^3}\right)\sum_{m=1}^{K}A_{k,m}^2\rho_m^2\\
&+(\sigma_k^F)^2(1-\rho)(-\mathrm{Li}_2(\rho)-\log(1-\rho))\sum_{m=1}^{K}\frac{A_{k,m}^2\rho_m}{\rho^2}\\
&+\frac{2A_{k,0}(\sigma_k^F)^2(1-\rho)}{\rho^2}(-\mathrm{Li}_2(\rho)-\log(1-\rho))\sum_{m=1}^{K}A_{k,m}\rho_m\\
&+\frac{(\sigma_k^F)^2}{\rho^3}[\rho+3(1-\rho)\log(1-\rho) +2(1-\rho)\,\mathrm{Li}_2(\rho)]\sum_{i=1}^{K}\sum_{j=1:j\neq i}^{K}A_{k,i}A_{k,j}\rho_i\rho_j
\end{aligned}$$

$$\mathbb{E}[(p_k^V(\vec{x}))^2] = (\sigma_k^V)^2 A_{k,0}^2 + \frac{2(\sigma_k^V)^2}{(1-\rho)^2}\sum_{m=1}^{K} A_{k,m}^2\rho_m^2 + \frac{(\sigma_k^V)^2}{(1-\rho)}\sum_{m=1}^{K} A_{k,m}^2\rho_m + \frac{2(\sigma_k^V)^2}{(1-\rho)}A_{k,0}\sum_{m=1}^{K} A_{k,m}\rho_m + \frac{2(\sigma_k^V)^2}{(1-\rho)^2}\sum_{i=1}^{K}\sum_{j=1:j\neq i}^{K} A_{k,i}A_{k,j}\rho_i\rho_j$$

$$\mathbb{E}[(p_k^L(\vec{x}))^2] = \frac{(\sigma_k^L)^2 A_{k,0}^2(1+\rho)}{(1-\rho)^2} + (\sigma_k^L)^2\sum_{m=1}^{K} A_{k,m}^2 \frac{2\rho_m(2+9\rho_m+3\rho_m\rho-\rho-\rho^2)}{(1-\rho)^4} + 2A_{k,0}(\sigma_k^L)^2\frac{2\rho+4}{(1-\rho)^3}\sum_{m=1}^{K} A_{k,m}\rho_m + \frac{(\sigma_k^L)^2 6(3+\rho)}{(1-\rho)^4}\sum_{i=1}^{K}\sum_{j=1:j\neq i}^{K} A_{k,i}A_{k,j}\rho_i\rho_j$$

The outline of the proof is provided in the appendix. We note that the proofs rely on further metrics such as $\mathbb{E}[x_i|\vec{x}|]$, $\mathbb{E}[x_i x_k|\vec{x}|^2]$ (higher moments), and $\mathbb{E}[x_i x_j/|\vec{x}|^2]$.

V. Simulation Results

To gain some qualitative insights, a server with QoS constraint $r_{\min} = 0.1$ bits/second and a single class of users ($\lambda_1 = 0.3$ packets/second, $\nu_1 = 1$ bit) is considered. The constants $A_{1,k}$ are determined as $A_{1,0} = \frac{2\lambda_1}{2\mu_1-\lambda_1}$, and $A_{1,1} = \frac{\lambda_1}{2\mu_1-\lambda_1}$ (see [8]). Figure 1 compares the second moment of user payments under pre-payment and post-payment mechanisms for the fixed rate pricing and VCG auctions. The discontinuity in the plots is attributed to the discontinuity in $n^* = \lfloor C/r_{\min}\rfloor$. For fixed rate pricing and VCG auctions, we observe that the pre-payment scheme has a smaller second moment than the post-payment scheme. The smaller second moment implies greater predictability in the revenue earned by the operator.

VI. Conclusion

In this paper, several performance measures for multiclass processor sharing have been computed that are of independent interest. Based on our preliminary work, pricing schemes that charge a user up-front are easier to implement, less volatile, generate the same average or long term revenue for the operator, and are thus preferable. Although we have only presented certain simulation results, in future work, we will discuss details of all pricing models.

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Appendix

Proof of Lemma 1: We have that $t(0) = \chi(\vec{0}) = \Phi(\vec{0})$.

$$\begin{aligned} t(n) &:= \sum_{\vec{x}:|\vec{x}|=n} \chi(\vec{x}) \\ &= \sum_{\vec{x}:|\vec{x}|=n} \frac{1}{C}\sum_{m=1}^{K}\Phi(\vec{x}-\vec{e}_m)\vec{\alpha}^{\vec{x}} \\ &= \sum_{m=1}^{K}\rho_m \sum_{\vec{x}:|\vec{x}|=n-1}\Phi(\vec{x})\vec{\alpha}^{\vec{x}} \\ &= \rho\cdot t(n-1). \end{aligned}$$

Also,

$$\sum_{\vec{x}}\chi(\vec{x}) = \sum_{n=0}^{\infty} t(n) = \frac{\Phi(\vec{0})}{1-\rho}.$$

■

Proof of Lemma 2: We start with

$$\begin{aligned} s_k(n) &= \sum_{\vec{x}:|\vec{x}|=n}\frac{x_k}{C}\sum_{m=1}^{K}\Phi(\vec{x}-\vec{e}_m)\vec{\alpha}^{\vec{x}} \\ &= \sum_{m=1}^{K}\rho_m\left[\sum_{\vec{x}:|\vec{x}|=n-1} x_k\chi(\vec{x}) + \sum_{\vec{x}:|\vec{x}|=n-1}(\vec{e}_m)_k\chi(\vec{x})\right] \\ &= \sum_{m=1}^{K}\rho_m s_k(n-1) + \sum_{m=1}^{K}\rho_m\sum_{\vec{x}:|\vec{x}|=n-1}(\vec{e}_m)_k\chi(\vec{x}). \end{aligned}$$

Or,

$$\begin{aligned} s_k(n) &= \rho s_k(n-1) + \rho_k t(n-1) \\ &= \rho s_k(n-1) + \rho_k\rho^{n-1}\Phi(\vec{0}). \end{aligned} \quad (17)$$

It is easily shown that the result is a solution to the recursion in (17). The first part of the result follows. Next,

$$\begin{aligned} \bar{s}_k(n) &= \rho_k\Phi(\vec{0})\sum_{m>n} m\rho^m \\ &= \rho_k\Phi(\vec{0})\frac{\rho^n}{1-\rho}\left(n+\frac{1}{1-\rho}\right). \end{aligned}$$

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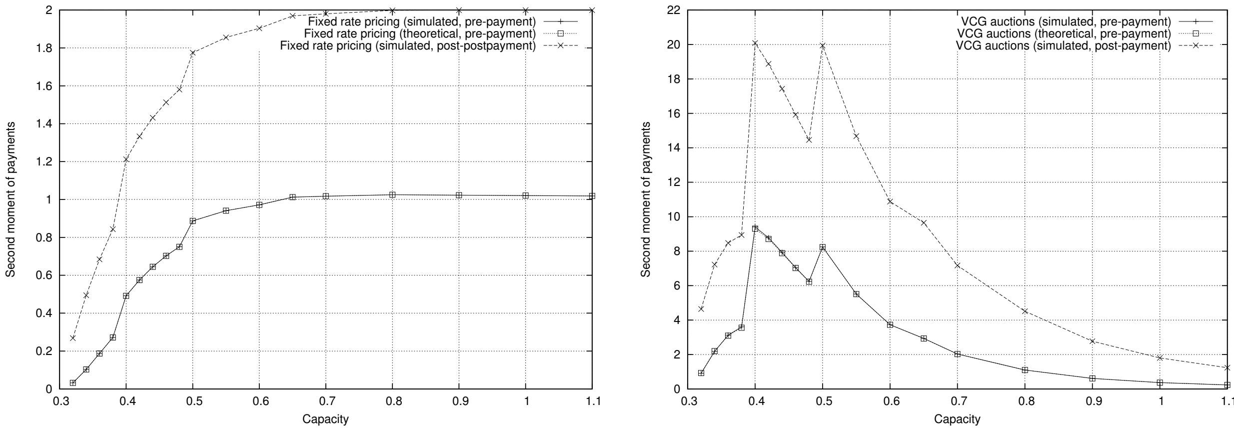
Proof of Lemma 3: The proof relies on establishing a recursive expression for $s_{i,j}(n)$. The expression in (6) is the



(a) Fixed-rate pricing

(b) VCG auctions

Fig. 1: Second moments from simulation and analysis.

solution of this recursion.

$$\begin{aligned}
s_{i,j}(n) &= \sum_{m=1}^{K} \frac{\alpha_m}{C} \sum_{\vec{x}:|\vec{x}|=n} x_i x_j \Phi(\vec{x}-\vec{e}_m)\vec{\alpha}^{\vec{x}-\vec{e}_m} \\
&= \sum_{m=1}^{K} \rho_m \sum_{\vec{y}:|\vec{y}|=n-1} (\vec{y}+\vec{e}_m)_i(\vec{y}+\vec{e}_m)_j \Phi(\vec{y})\vec{\alpha}^{\vec{y}} \\
&= \rho s_{i,j}(n-1) + \rho_j s_i(n-1) \\
&\quad + \rho_i s_j(n-1) + \mathbf{1}_{(i=j)}\rho_i t(n-1).
\end{aligned}$$

Note that $s_{i,j}(0) = 0$ for any i, j and that $s_{i,j}(1) = 0$ if $i \neq j$. ■

Proof of Proposition 1:

$$\begin{aligned}
u(n) &= \sum_{\vec{x}:|\vec{x}|>n} \left(|\vec{x}| - \frac{1}{2}\right)\pi(\vec{x}) \\
&= \sum_{\vec{x}:|\vec{x}|>n} |\vec{x}|\pi(\vec{x}) - \frac{1}{2}\sum_{\vec{x}:|\vec{x}|>n} \pi(\vec{x}) \\
&= \sum_{\vec{x}:|\vec{x}|>n} \frac{(x_1+\cdots+x_K)\chi(\vec{x})}{\sum_{\vec{y}}\chi(\vec{y})} - \frac{1}{2}\sum_{\vec{x}:|\vec{x}|>n} \frac{\chi(\vec{x})}{\sum_{\vec{y}}\chi(\vec{y})}.
\end{aligned}$$

Using $\sum_{\vec{x}} \chi(\vec{x}) = \sum_{n\geq 0} t(n)$,

$$u(n) = \frac{\sum_{k=1}^{K} \bar{s}_k(n)}{\sum_{m\geq 0} t(m)} - \frac{\sum_{m>n} t(m)}{2\sum_{m\geq 0} t(m)}.$$

Using Lemma 1 and Lemma 2, we get

$$u(n) = \rho^{n+1}\left(n + \frac{1}{1-\rho}\right) - \frac{\rho^{n+1}}{2}.$$

■

Proof of Proposition 2:

$$\begin{aligned}
R_V(\vec{x}) &= |\vec{x}|(|\vec{x}|-1)\log\left(\frac{|\vec{x}|}{|\vec{x}|-1}\right) \\
&= |\vec{x}|(|\vec{x}|-1)\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}\left(\frac{1}{|\vec{x}|-1}\right)^n
\end{aligned}$$

Subtracting $(|\vec{x}| - 1/2)$ from both sides,

$$\begin{aligned}
R_V(\vec{x}) - \left(|\vec{x}| - \frac{1}{2}\right) &= \frac{1}{2} - \frac{|\vec{x}|}{2(|\vec{x}|-1)} + \frac{|\vec{x}|}{3(|\vec{x}|-1)^2} - \dots \\
&= -\frac{1}{2(|\vec{x}|-1)} + \sum_{m=3}^{\infty} \frac{(-1)^{m-1}|\vec{x}|}{m(|\vec{x}|-1)^{m-1}}.
\end{aligned}$$

This proves the first part. For the second part, rewrite the m-th term above as

$$\frac{|\vec{x}|}{m(|\vec{x}|-1)^{m-1}} = \frac{1}{m(|\vec{x}|-1)^{m-2}} + \frac{1}{m(|\vec{x}|-1)^{m-1}}.$$

Simplifying gives

$$\begin{aligned}
R_V - \left(|\vec{x}| - \frac{1}{2}\right) &= \sum_{m=1}^{\infty} \frac{(-1)^m}{(m+1)(m+2)(|\vec{x}|-1)^m} \\
&= \sum_{m=1,3,\dots} \frac{(-1)^m}{(m+2)(|\vec{x}|-1)^m}\left[\frac{1}{m+1}\right. \\
&\qquad \left. - \frac{1}{(m+3)(|\vec{x}|-1)}\right] \\
&< 0
\end{aligned}$$

since $\frac{1}{m+1} - \frac{1}{(m+3)(|\vec{x}|-1)} > 0$ for $|\vec{x}| > 1$. ■

Proof of Proposition 3: Let the system be in state $\vec{x}$. The mean revenue under fixed rate pricing is given by

$$\begin{aligned}
\bar{R}_F &= \sum_{\vec{x}:1\leq|\vec{x}|\leq n^*} \beta C\pi(\vec{x}) \\
&= \beta C \frac{1}{\sum_{\vec{x}}\chi(\vec{x})} \sum_{\vec{x}:1\leq|\vec{x}|\leq n^*} \chi(\vec{x}) \\
&= \beta C \frac{\sum_{n=1}^{n^*} t(n)}{\sum_{n=0}^{\infty} t(n)}.
\end{aligned}$$

Using Lemma 1,

$$\bar{R}_F = \beta C \frac{1-\rho}{\Phi(\vec{0})} \frac{\Phi(\vec{0})(\rho - \rho^{n^*+1})}{1-\rho} = \beta C\rho(1-\rho^{n^*}).$$

The mean revenue under VCG pricing is

$$\begin{aligned}\bar{R}_V &= \sum_{\vec{x}:2\leq|\vec{x}|\leq n^*}\left(|\vec{x}|-\frac{1}{2}\right)\pi(\vec{x})\\ &= u(1)-u(n^*).\end{aligned}$$

The result for $\bar{R}_V$ follows by simplification. The mean revenue under congestion-based pricing is given by

$$\begin{aligned}\bar{R}_L &= \sum_{\vec{x}:1\leq|\vec{x}|\leq n^*}\frac{|\vec{x}|^2}{C}\pi(\vec{x})\\ &= \frac{1}{C}\sum_{\vec{x}:1\leq|\vec{x}|\leq n^*}|\vec{x}|^2\frac{\chi(\vec{x})}{\sum_{\vec{y}}\chi(\vec{y})}\\ &= \frac{1}{C}\frac{\sum_{1\leq n\leq n^*}v(n)}{\sum_{n\geq 0}t(n)}.\end{aligned}$$

Using $v(n)=n^2t(n)$ and Lemma 1,

$$\bar{R}_L = \frac{1-\rho}{C}\sum_{n=1}^{n^*}n^2\rho^n.$$

The following identity (for $\rho<1$) simplifies the summation.

$$\begin{aligned}\sum_{n=1}^{m}n^2\rho^n = \frac{\rho}{(1-\rho)^3}[1+\rho-\rho^m(m^2\rho^2\\ -(2m^2+2m-1)\rho+(m+1)^2)]\end{aligned}$$

■

Proof of Proposition 4: To evaluate the mean payment by a class k user under fixed rate pricing, consider the following integral where A_k is the arrival process for class k users and W_0^k is the random variable denoting the sojourn time of the class k arrival at time 0.

$$\bar{c}_k^F = \mathbb{E}_{A_k}\left[\int_0^{W_0^k}\frac{\beta C}{|\vec{x}(t)|}\mathbf{1}_{(1\leq|\vec{x}(t)|\leq n^*)}dt\right]$$

Applying the Swiss Army formula (see [9]),

$$\begin{aligned}\bar{c}_k^F &= \frac{\beta C}{\lambda_k}\mathbb{E}\left[\frac{x_k}{|\vec{x}|}\mathbf{1}_{(1\leq|\vec{x}|\leq n^*)}\right]\\ &= \frac{\beta C}{\lambda_k}\sum_{n=1}^{n^*}\sum_{\vec{x}:|\vec{x}|=n}\frac{x_k}{n}\pi(\vec{x})\\ &= \frac{\beta C}{\lambda_k}\frac{1}{\sum_{n=0}^{\infty}t(n)}\sum_{n=1}^{n^*}g_k(n)\\ &= \nu_k\beta\rho(1-\rho^{n^*}).\end{aligned}$$

Similarly, for VCG pricing, the mean payment for a class k user is given by

$$\bar{c}_k^V = \mathbb{E}_{A_k}\left[\int_0^{W_0^k}\left(1-\frac{1}{2|\vec{x}(t)|}\right)\mathbf{1}_{(2\leq|\vec{x}(t)|\leq n^*)}dt\right].$$

Again, using the Swiss Army formula,

$$\begin{aligned}\bar{c}_k^V &= \frac{1}{\lambda_k}\mathbb{E}\left[\left(1-\frac{1}{2|\vec{x}|}\right)\mathbf{1}_{(2\leq|\vec{x}|\leq n^*)}\right]\\ &= \frac{1}{\lambda_k}\mathbb{E}[x_k\mathbf{1}_{(2\leq|\vec{x}|\leq n^*)}]\\ &\quad -\frac{1}{\lambda_k}\mathbb{E}\left[\frac{x_k}{2|\vec{x}|}\mathbf{1}_{(2\leq|\vec{x}|\leq n^*)}\right].\end{aligned} \tag{18}$$

Let J_1 and J_2 be the first and the second term respectively in (18). Then,

$$\begin{aligned}J_1 &= \frac{1}{\lambda_k}\sum_{n=2}^{n^*}\sum_{\vec{x}:|\vec{x}|=n}x_k\pi(\vec{x})\\ &= \frac{1-\rho}{\lambda_k\Phi(\vec{0})}\sum_{n=2}^{n^*}s_k(n)\\ &= \frac{\nu_k(1-\rho)}{C}\sum_{n=2}^{n^*}n\rho^{n^*-1},\end{aligned}$$

and,

$$\begin{aligned}J_2 &= \frac{1}{2\lambda_k}\sum_{n=2}^{n^*}\sum_{\vec{x}:|\vec{x}|=n}\frac{x_k}{|\vec{x}|}\pi(\vec{x})\\ &= \frac{1-\rho}{2\Phi(\vec{0})\lambda_k}\sum_{n=2}^{n^*}g_k(n)\\ &= \frac{\nu_k\rho}{2C}(1-\rho^{n^*-1}).\end{aligned}$$

Using the identity

$$\sum_{n=1}^{m}n\rho^{n-1} = \frac{1-\rho^{m+1}-(m+1)(1-\rho)\rho^m}{(1-\rho)^2},$$

and simplifying provides the required result. For congestion-based pricing, the mean payment by a class k user is given by

$$\bar{c}_k^L = \mathbf{E}_{A_k}\left[\int_0^{W_0^k}\frac{|\vec{x}(t)|}{C}\mathbf{1}_{(1\leq|\vec{x}(t)|\leq n^*)}dt\right].$$

Applying the Swiss Army formula gives,

$$\begin{aligned}\bar{c}_k^L &= \frac{1}{\lambda_k C}\mathbb{E}[x_k|\vec{x}|\mathbf{1}_{(1\leq|\vec{x}|\leq n^*)}]\\ &= \frac{1}{\lambda_k C}\sum_{n=1}^{n^*}\sum_{\vec{x}:|\vec{x}|=n}x_k n\pi(\vec{x})\\ &= \frac{1-\rho}{\lambda_k C\Phi(\vec{0})}\sum_{n=1}^{n^*}ns_k(n)\\ &= \frac{\nu_k(1-\rho)}{C^2}\sum_{n=1}^{n^*}n^2\rho^{n-1},\end{aligned}$$

which shows the required result. ■

Proof of Proposition 6: Suppose a class k arrival sees the system state as $\vec{x}$ on arrival. The fixed rate, pre-payment price charged is

$$p_k^F(\vec{x}) = \frac{\sigma_k^F}{|\vec{x}+\vec{e}_k|}\left(A_{k,0} + \sum_{m=1}^{K} A_{k,m}x_m\right).$$

It is required that the mean payment by a class k user equal $\bar{c}_k^F$, i.e.,

$$\mathbb{E}[p_k^F(\vec{x})] = \bar{c}_k^F. \tag{19}$$

Starting with the left hand side (LHS) of (19),

$$\begin{aligned} LHS &= \sum_{\vec{x}} \frac{\sigma_k^F}{|\vec{x}+\vec{e}_k|}\pi(\vec{x})\left(A_{k,0} + \sum_{m=1}^{K} A_{k,m}x_m\right) \\ &= \sigma_k^F(1-\rho)\left[\frac{A_{k,0}}{\rho}\log\frac{1}{1-\rho}\right. \\ &\quad \left. + \left(\frac{\frac{\rho}{1-\rho} - \log\frac{1}{1-\rho}}{\rho^2}\right)\sum_{m=1}^{K} A_{k,m}\rho_m\right] \end{aligned}$$

Equating this to $\bar{c}_k^F$ gives σ_k^F. Similarly, under VCG pricing,

$$p_k^V(\vec{x}) = \sigma_k^V\left(A_{k,0} + \sum_{m=1}^{K} A_{k,m}x_m\right),$$

and

$$\begin{aligned} \mathbb{E}[p_k^V(\vec{x})] &= \sigma_k^V \sum_{\vec{x}}\left(A_{k,0} + \sum_{m=1}^{K} A_{k,m}x_m\right)\pi(\vec{x}) \\ &= \sigma_k^V\,\mathbb{E}[W_k] \\ &= \sigma_k^V\frac{\nu_k}{C(1-\rho)}. \end{aligned}$$

Equating this to $\bar{c}_k^V$ gives σ_k^V. Last, under congestion-based pricing,

$$p_k^L(\vec{x}) = \sigma_k^L|\vec{x}+\vec{e}_k|\left(A_{k,0} + \sum_{m=1}^{K} A_{k,m}x_m\right).$$

Taking the expectation gives

$$\begin{aligned} \mathbb{E}[p_k^L(\vec{x})] &= \sigma_k^L\sum_{\vec{x}}|\vec{x}+\vec{e}_k|\left(A_{k,0} + \sum_{m=1}^{K} A_{k,m}x_m\right)\pi(\vec{x}) \\ &= \frac{\sigma_k^L}{1-\rho}\left[A_{k,0} + \frac{2}{1-\rho}\sum_{m=1}^{K} A_{k,m}\rho_m\right], \end{aligned}$$

and equating this to $\bar{c}_k^L$ gives σ_k^L.

■

Proof of Proposition 7: The steps for deriving the second moment under congestion-based pricing are outlined here. The proof for the other two pricing models is similar.

$$\begin{aligned} &\mathbb{E}[(p_k^L(\vec{x}))^2] = \sum_{\vec{x}}(p_k^L(\vec{x}))^2\pi(\vec{x}) \\ &= (\sigma_k^L)^2\sum_{\vec{x}}|\vec{x}+\vec{e}_k|^2\left(A_{k,0} + \sum_{m=1}^{K} A_{k,m}x_m\right)^2\pi(\vec{x}) \\ &= (\sigma_k^L)^2\sum_{\vec{x}}|\vec{x}+\vec{e}_k|^2 A_{k,0}^2\pi(\vec{x}) \\ &\quad + (\sigma_k^L)^2\sum_{\vec{x}}|\vec{x}+\vec{e}_k|^2\sum_{m=1}^{K} A_{k,m}^2x_m^2\pi(\vec{x}) \\ &\quad + (\sigma_k^L)^2 2\sum_{\vec{x}}|\vec{x}+\vec{e}_k|^2 A_{k,0}\sum_{m=1}^{K} A_{k,m}x_m\pi(\vec{x}) \\ &\quad + (\sigma_k^L)^2\sum_{\vec{x}}|\vec{x}+\vec{e}_k|^2\sum_{i=1}^{K}\sum_{j=1:j\neq i}^{K} A_{k,i}A_{k,j}x_ix_j\pi(\vec{x}) \\ &:= S_1 + S_2 + S_3 + S_4 \end{aligned}$$

$$\begin{aligned} S_1 &= (\sigma_k^L)^2\sum_{\vec{x}}|\vec{x}+\vec{e}_k|^2 A_{k,0}^2\pi(\vec{x}) \\ &= \frac{(\sigma_k^L)^2A_{k,0}^2(1-\rho)}{\Phi(\vec{0})}\sum_{\vec{x}}(|\vec{x}|+1)^2\chi(\vec{x}) \\ &= \frac{(\sigma_k^L)^2A_{k,0}^2(1+\rho)}{(1-\rho)^2} \end{aligned}$$

$$\begin{aligned} S_2 &= (\sigma_k^L)^2\sum_{\vec{x}}|\vec{x}+\vec{e}_k|^2\sum_{m=1}^{K} A_{k,m}^2x_m^2\pi(\vec{x}) \\ &= \frac{(\sigma_k^L)^2(1-\rho)}{\Phi(\vec{0})}\sum_{m=1}^{K} A_{k,m}^2\sum_{n=0}^{\infty}(n+1)^2 s_{m,m}(n) \\ &= 2(\sigma_k^L)^2\sum_{m=1}^{K} A_{k,m}^2\frac{\rho_m(2+9\rho_m+3\rho_m\rho-\rho-\rho^2)}{(1-\rho)^4}. \end{aligned}$$

$$\begin{aligned} S_3 &= (\sigma_k^L)^2\sum_{\vec{x}}|\vec{x}+\vec{e}_k|^2 2A_{k,0}\sum_{m=1}^{K} A_{k,m}x_m\pi(\vec{x}) \\ &= \frac{2A_{k,0}(\sigma_k^L)^2(1-\rho)}{\Phi(\vec{0})}\sum_{m=1}^{K} A_{k,m}\sum_{n=0}^{\infty}(n^2+2n+1)s_m(n) \\ &= 2A_{k,0}(\sigma_k^L)^2\frac{(2\rho+4)}{(1-\rho)^3}\sum_{m=1}^{K} A_{k,m}\rho_m. \end{aligned}$$

$$\begin{aligned} S_4 &= (\sigma_k^L)^2\sum_{\vec{x}}|\vec{x}+\vec{e}_k|^2\sum_{i=1}^{K}\sum_{j=1:j\neq i}^{K} A_{k,i}A_{k,j}x_ix_j\pi(\vec{x}) \\ &= \frac{(\sigma_k^L)^2(1-\rho)}{\Phi(\vec{0})}\sum_{i=1}^{K}\sum_{j=1:j\neq i}^{K} A_{k,i}A_{k,j}\sum_{n=0}^{\infty}(n+1)^2 s_{i,j}(n) \\ &= \frac{6(\sigma_k^L)^2(3+\rho)}{(1-\rho)^4}\sum_{i=1}^{K}\sum_{j=1:j\neq i}^{K} A_{k,i}A_{k,j}\rho_i\rho_j. \end{aligned}$$

Combining S_1, S_2, S_3 and S_4 gives the result. ■